

The maximum number of cliques in graphs without long cycles

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Abstract

The Erdős–Gallai Theorem states that for $k \geq 3$ every graph on n vertices with more than $\frac{1}{2}(k-1)(n-1)$ edges contains a cycle of length at least k . Kopylov proved a strengthening of this result for 2-connected graphs with extremal examples $H_{n,k,t}$ and $H_{n,k,2}$. In this note, we generalize the result of Kopylov to bound the number of s -cliques in a graph with circumference less than k . Furthermore, we show that the same extremal examples that maximize the number of edges also maximize the number of cliques of any fixed size. Finally, we obtain the extremal number of s -cliques in a graph with no path on k -vertices.

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1 Introduction

In [1], Erdős and Gallai determined $ex(n, P_k)$, the maximum number of edges in an n -vertex graph that does not contain a copy of the path on k vertices, P_k . This result was a corollary of the following theorem:

Theorem 1.1 (Erdős and Gallai [1]). *Let G be an n -vertex graph with more than $\frac{1}{2}(k-1)(n-1)$ edges, $k \geq 3$. Then G contains a cycle of length at least k .*

To obtain the result for paths, suppose G is an n -vertex graph with no copy of P_k . Add a new vertex v adjacent to all vertices in G , and let this new graph be G' . Then G' is an $n+1$ -vertex graph with no cycle of length $k+1$ or longer, and so $e(G) + n = e(G') \leq \frac{1}{2}(k)(n)$ edges.

Corollary 1.2 (Erdős and Gallai [1]). *Let G be an n -vertex graph with more than $\frac{1}{2}(k-2)n$ edges, $k \geq 2$. Then G contains a copy of P_k .*

Both results are sharp with the following extremal examples: for Theorem 1.1, when $k-2$ divides $n-1$, take any connected n -vertex graph whose blocks are cliques of order $k-1$. For Corollary 1.2, when $k-1$ divides $n-1$, take the n -vertex graph whose connected components are cliques of order $k-1$.

There have been several alternate proofs and sharpenings of the Erdős–Gallai theorem including results by Woodall [7], Lewin [10], Faudree and Schelp [8, 9], and Kopylov [2] – see [11] for further details.

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The strongest version was that of Kopylov who improved the Erdős–Gallai bound for 2-connected graphs. To state the theorem, we first introduce the family of extremal graphs.

Fix $k \geq 4$ $n \geq k$, $\frac{k}{2} > a \geq 1$. Define the n -vertex graph $H_{n,k,a}$ as follows. The vertex set of $H_{n,k,a}$ is partitioned into three sets A, B, C such that $|A| = a$, $|B| = n - k + a$ and $|C| = k - 2a$ and the edge set of $H_{n,k,a}$ consists of all edges between A and B together with all edges in $A \cup C$.

Note that when $a \geq 2$, $H_{n,k,a}$ is 2-connected, has no cycle of length at least k , and $e(H_{n,k,a}) = \binom{k-a}{2} + (n - k + a)a$.

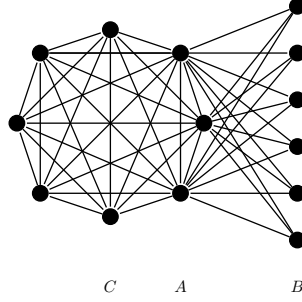


Figure 1: $H_{14,11,3}$

Definition. Let $f_s(n, k, a) := \binom{k-a}{s} + (n - k + a)\binom{a}{s-1}$, where $f_2(n, k, a) = e(H_{n,k,a})$.

By considering the second derivative, one can check that $f_s(n, k, a)$ is convex in a in the domain $[1, \lfloor (k-1)/2 \rfloor]$, thus it attains its maximum at one of the endpoints $a = 1$ or $a = \lfloor (k-1)/2 \rfloor$.

Theorem 1.3 (Kopylov [2]). *Let $n \geq k \geq 5$ and let $t = \lfloor \frac{k-1}{2} \rfloor$. If G is a 2-connected n -vertex with*

$$e(G) \geq \max\{f_2(n, k, 2), f_2(n, k, t)\},$$

then either G has a cycle of length at least k , or $G = H_{n,k,t}$, or $G = H_{n,k,2}$.

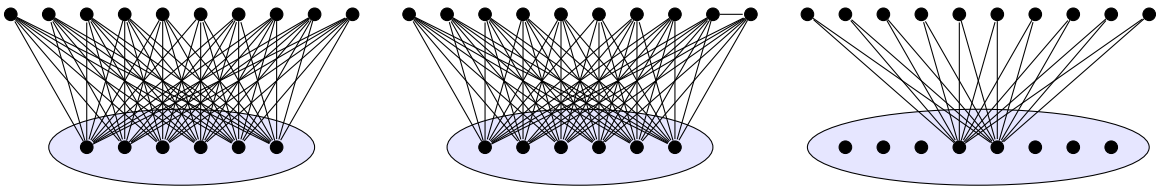


Figure 2: $H_{n,k,t}(k = 2t + 1)$, $H_{n,k,t}(k = 2t + 2)$, $H_{n,k,2}$; ovals denote complete subgraphs.

This theorem also implies Theorem 1.1 by applying induction to each block of the graph.

We consider a generalized Turán-type problem. Fix graphs T and H , and define the function $ex(n, T, H)$ to be the maximum number of (unlabeled) copies of T in an H -free graph on n vertices. When $T = K_2$, we have the usual extremal number $ex(n, T, H) = ex(n, H)$.

Some notable papers that study the $ex(n, T, H)$ function for different combinations of T and H include [3] by Erdős who proved that for $s \leq r$, among graphs that forbid K_{r+1} , the balanced complete r -partite graph also maximizes the number of copies of K_s , [4] by Alon and Shikhelman who proved $ex(n, K_s, K_{r,t}) = \Theta(n^{s - \binom{s}{2}/r})$ for certain values of r and t , [5] by Hatami, Hladký,

Král', Norine, and Razborov who proved $ex(n, C_5, K_3) = (n/5)^5$ using the method of flag algebras, and [6] by Füredi and Özkahya who proved an upper bound for the number of triangles in a graph without cycles of a fixed size (in terms of the extremal number of the cycles).

In this note, we give an upper bound for the number of s -cliques in a graph without cycles of length at least k (i.e., circumference less than k). We also obtain $ex(n, K_s, P_k)$.

Definition. For $s \geq 2$, let $N_s(G)$ denote the number of unlabeled copies of K_s in G , e.g., $N_2(G) = e(G)$.

Our main result is a generalization of Kopylov's result, Theorem 1.3. In particular, we show that the same extremal examples that maximize the number of edges among n -vertex 2-connected graphs with circumference less than k also maximize the number of cliques of any size. Our main results are the following.

Theorem 1.4. *Let $n \geq k \geq 5$ and let $t = \lfloor \frac{k-1}{2} \rfloor$. If G is a 2-connected n -vertex graph with circumference less than k , then*

$$N_s(G) \leq \max\{f_s(n, k, 2), f_s(n, k, t)\}.$$

Again, this theorem is sharp with the same extremal examples $H_{n,k,t}$ and $H_{n,k,2}$.

This theorem implies the cliques version of Theorem 1.1.

Corollary 1.5. *Let $n \geq k \geq 3$ and let $t = \lfloor \frac{k-1}{2} \rfloor$. If G is an n -vertex graph with circumference less than k , then*

$$N_s(G) \leq \frac{n-1}{k-2} \binom{k-1}{s}.$$

Unlike the edges case, Theorem 1.4 unfortunately does not easily imply $ex(n, K_s, P_k)$. However, a Kopylov-style argument very similar to the proof of Theorem 1.4 gives the result for paths.

Theorem 1.6. *Let $n \geq k$ and let G be an n -vertex connected graph with no path on k vertices. Let $t = \lfloor (k-2)/2 \rfloor$. Then $N_s(G) \leq \max\{f_s(n, k-1, t), f_s(n, k-1, 1)\}$.*

We have sharpness examples $H_{n,k-1,t}$ and $H_{n,k-1,1}$. Finally, using induction on the number of components gives the following result:

Corollary 1.7. $ex(n, K_s, P_k) = \frac{n}{k-1} \binom{k-1}{s}.$

And the same extremal examples for Corollary 1.2 apply.

The proofs for Corollary 1.5, Theorem 1.6, and Theorem 1.7 are given in Section 3 of this paper. We first prove Theorem 1.4.

2 Proof of Theorem 1.4

Let G be a edge-maximal counterexample. Then G is k -closed, i.e., adding any addition edge to G creates a cycle of length at least k . In particular, for any nonadjacent vertices x and y of G , there exists a path of at least $k-1$ edges between x and y . We will use the following lemma:

Lemma 2.1 (Kopylov [2]). *Let G be a 2-connected n -vertex graph with a path P of m edges with endpoints x and y . Then G contains a cycle of length at least $\min\{m+1, d_P(x) + d_P(y)\}$.*

Our first goal is to show that G contains a large “core”, i.e., a subgraph with large minimum degree. For this, we use the notion of *disintegration*.

Definition: For a natural number α and a graph G , the α -*disintegration* of a graph G is the process of iteratively removing from G the vertices with degree at most α until the resulting graph has minimum degree at least $\alpha + 1$ or is empty. This resulting subgraph $H = H(G, \alpha)$ will be called the α -*core* of G . It is well known that $H(G, \alpha)$ is unique and does not depend on the order of vertex deletion.

Let $H(G, t)$ denote the t -core of G , i.e., the resulting graph of applying t -disintegration to G . We claim that

$$H(G, t) \text{ is nonempty.}$$

Suppose $H(G, t)$ is empty. In the disintegration process, every time a vertex of degree at most t is removed, we delete at most $\binom{t}{s-1}$ copies of K_s . For the last $\ell \leq t$ vertices, we remove at most $\binom{\ell-1}{s-1}$ copies of K_s with each deletion. Thus

$$\begin{aligned} N_s(G) &\leq (n-t) \binom{t}{s-1} + \binom{t-1}{s-1} + \binom{t-2}{s-1} + \dots + \binom{0}{s-1} \\ &= (n-t) \binom{t}{s-1} + \binom{t}{s} \\ &= (n-(t+1)) \binom{t}{s-1} + \binom{t+1}{s} \\ &\leq f_s(n, k, t), \end{aligned}$$

a contradiction.

Therefore $H(G, t)$ is nonempty. Next we show that

$$H(G, t) \text{ is a complete graph.}$$

If there exists a nonedge of $H(G, t)$, then in G , there is a cycle of length at least $k-1$ edges with these vertices as its endpoints. Among all nonadjacent pairs of vertices in $H(G, t)$, choose x, y such that there is a longest path P with endpoints x and y . By maximality of P , all neighbors of x in $H(G, t)$ lie in P . Similar for y . Hence, by Lemma 2.1, G has a cycle of length at least $\min\{k, d_P(x) + d_P(y)\} = \min\{k, 2(t+1)\} = k$, a contradiction.

Now let $r = |V(H(G, t))|$. Each vertex in $H(G, t)$ has degree at least $t+1$, so $r \geq t+2$. Also, if $r \geq k-1$, because G is 2-connected, we can extend a path on r vertices of $H(G, t)$ to a cycle of length at least $r+1 \geq k$, a contradiction. Therefore $t+2 \leq r \leq k-2$. In particular, $k-r \leq t$. Apply a weaker $(k-r)$ -disintegration to G , and let $H(G, k-r)$ be the resulting graph. Then $H(G, t) \subseteq H(G, k-r)$.

If $H(G, t) = H(G, k-r)$, then $N_s(G) \leq \binom{r}{s} + (n-r) \binom{k-r}{s-1} = f_s(n, k, k-r) \leq f_s(n, k, 2)$. Therefore, $H(G, t)$ is a proper subgraph of $H(G, k-r)$, and there must be a nonedge between a vertex in $H(G, t)$

and a vertex in $H(G, k-r)$. Among all such pairs, choose $x \in H(G, t)$ and $y \in H(G, k-r)$ to have a longest path between them. Then G contains a cycle of length at least $\min\{k, r-1+k-r+1\} = k$, a contradiction. \square

3 Proof of Corollary 1.5, Theorem 1.6, and Corollary 1.7

Define $g_s(n, k) = \frac{n-1}{k-2} \binom{k-1}{s}$ and $t = \lfloor \frac{k-1}{2} \rfloor$. One can check that when $n \geq k$,

$$g_s(n, k) \geq \max\{f_s(n, k, t), f_s(n, k, 2)\}.$$

Proof of Corollary 1.5. Fix a graph G on n vertices with circumference less than k . We induct on the number of blocks of G . If G is a block, i.e., 2-connected, then either $n \leq k-1$, and so $N_s(G) \leq \binom{|V(G)|}{s} \leq g_s(n, k)$, or $n \geq k$, and so by Theorem 1.4, $N_s(G) \leq \max\{f_s(n, k, t), f_s(n, k, 2)\} \leq g_s(n, k)$. Otherwise, let $G = B_1 \cup B_2$ where B_1 is a block, and $B_2 = G - B_1$. Apply the induction hypothesis to B_1 and B_2 to obtain

$$N_s(G) = N_s(B_1) + N_s(B_2) \leq g_s(|B_1|, k) + g_s(n - |B_1|, k) = g_s(n, k).$$

\square

The proof of Theorem 1.6 follows the same steps as the proof of Theorem 1.4. As some details here will be omitted to prevent repetition, it is advised that the reader first reads the proof of Theorem 1.4.

Proof of Theorem 1.6. Suppose for contradiction that $N_s(G) > \max\{f_s(n, k-1, t), f_s(n, k-1, 1)\}$. Let G_0 be the graph obtained by adding a dominating vertex v_0 adjacent to all of $V(G)$. Then G_0 is 2-connected, has $n+1$ vertices, and contains no cycle of length $k+1$ or greater. Let G' be the $k+1$ -closure of G_0 (i.e., add edges to G_0 until any additional edge creates a cycle of length at least $k+1$). Denote by $N'_s(G')$ the number of K_s 's in G' that do not contain v_0 . Thus $N'_s(G') \geq N'_s(G_0) = N_s(G)$. Apply $t+1$ -disintegration to G' , where if necessary, we delete v_0 last. If $H(G', t+1)$ is empty, then at the time of deletion each vertex has at most t neighbors that are not v_0 . Hence

$$N'_s(G') \leq (n - (t+1)) \binom{t}{s-1} + \binom{t+1}{s} \leq f_s(n, k-1, t),$$

a contradiction.

The same argument as in the proof of Theorem 1.4 also shows that $H(G', t+1)$ is a complete graph, otherwise there would be a cycle of length at least $2(t+2) \geq (k-1) + 2$ in G' . Note that v_0 must be contained in $H(G', t+1)$ as it is adjacent to all vertices in G' . Set $|V(H(G', t+1))| = r$ where $t+3 \leq r \leq k-1$ (and so $k-r \geq 1$). In particular, $(k+1) - r < t-1$. Apply a weaker $k+1-r$ disintegration to G' . If $H(G', t+1) \neq H(G', k+1-r)$, then again we can find a cycle of length at least $(r-1) + k + 2 - r = k+1$. Otherwise, suppose $H(G', t+1) = H(G', k+1-r)$. In $H(G', t+1)$, the number of s -cliques that do not include v_0 is $\binom{r-1}{s}$, and in $V(G) - V(H(G', k+1-r))$, every

vertex had at most $k - r$ neighbors that were not v_0 at the time of its deletion. We have

$$N'_s(G') \leq \binom{r-1}{s} + (n+1-r) \binom{k-r}{s-1} = f_s(n, k-1, k-r) \leq f_s(n, k-1, 1),$$

a contradiction. □

Define $h_s(n, k) = \frac{n}{k-1} \binom{k-1}{s}$, and note that when $n \geq k$,

$$h_s(n, k) \geq \max\{f_s(n, k-1, t), f_s(n, k-1, 1)\}.$$

Proof of Corollary 1.7. We induct on the number of components in G . If G is connected, then either $n \leq k-1$, in which case $N_s(G) \leq \binom{|V(G)|}{s} \leq h_s(n, k)$, or $n \geq k$ and $N_s(G) \leq \max\{f_s(n, k-1, t), f_s(n, k-1, 1)\} \leq h_s(n, k)$. Otherwise if G is not connected, let C_1 be a component of G . Then $N_s(G) = N_s(C_1) + N_s(G - C_1) \leq h_s(|C_1|, k) + h_s(n - |C_1|, k) = h_s(n, k)$. □

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